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Self-Projective Rational Curves of the Fourth and Fifth Orders.*

BY R. M. WINGER.

§ 1. *Introduction.*

In the study of a rational quintic, first pointed out by Cayley, through the consideration of special cases I was led to a curve which is invariant under a group of six collineations.† The question naturally arose as to other curves unaltered by the group, and this was later extended to the problem of the discovery of rational curves invariant under other finite collineation groups. Such curves have been called *self-projective*. Many of these have found their way into the literature, but usually quite apart from or incidental to their connection with the groups. And while the general quartics and quintics have been discussed by Ciani‡ and Snyder§ respectively, the rational case has not been systematically treated. Neither is this phase of the question to be ignored as a special case of the work cited above, for the order of imposing the conditions materially affects the results.

As the variety of self-projective curves is rather large, some criterion of classification had to be adopted. The purpose of this paper is to present in canonical form *all of the projectively distinct types of the most general rational curves of the fourth and fifth orders invariant under the different finite collineation groups.*

Of fundamental importance in the study for rational curves is the

THEOREM. *To every ternary collineation of the points of the curve there is a corresponding binary collineation of the parameter, and conversely.*

Both of these must leave the curve unaltered if it is to be self-projective. The groups to be considered, then, are those generated by the regular body groups : ||

cyclic	g_n ,
dihedral	g_{2n} ,
tetrahedral	g_{12} ,
octahedral	g_{24} ,
icosahedral	g_{60} .

* Read before the American Mathematical Society, January 2, 1913.

† Discussed, § 14.

‡ *Istituto Lombardo Rendiconti*, Series II, Vol. XXXIII, p. 1170.

§ AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX.

|| Klein, "Ikosaeder."

It is no condition on a conic to admit any of these groups.

Again, *every member of the syzygetic pencil of rational cubics*

$$x_0^3 + x_1^3 + \lambda x_0 x_1 x_2 = 0$$

is invariant under a dihedral g_6 .

The cubic with a cusp,

$$x_0 = t^3, \quad x_1 = t^2, \quad x_2 = 1,$$

is unchanged by the substitution

$$t = \alpha t' \quad \text{or} \quad \begin{cases} x'_0 = \alpha^3 x_0, \\ x'_1 = \alpha^2 x_1, \\ x'_2 = x_2. \end{cases}$$

That is, *the cuspidal cubic is invariant under a one-parameter group.*

I. THE RATIONAL QUARTIC.

§ 2. *The Stahl Binary Sextic.*

The first case of considerable interest, then, is the rational quartic, ρ^4 , which will not admit a group without restriction.

The labor is greatly facilitated by reason of the existence of a *fundamental sextic* (binary) of which line sections are second polars and upon which the invariant theory of the curve depends.*

Bolza† has tabulated the canonical forms of the non-singular binary sextics with linear transformations into themselves. He exhibits the following types with the corresponding groups:

(1)	$t^6 + \alpha t^4 + \beta t^2 + 1$	cyclic	g_2 ,
(2)	$t(t^5 + 1)$	cyclic	g_5 ,
(3)	$t(t^4 + \alpha t^2 + 1)$	dihedral	g_4 ,
(4)	$t^6 + \alpha t^3 + 1$	dihedral	g_6 ,
(5)	$t^6 + 1$	dihedral	g_{12} ,
(6)	$t(t^4 + 1)$	octahedral	g_{24} .

We supplement this with the singular form ‡

$$(7) \quad t^2(t^4 + 1) \quad \text{cyclic} \quad g_4.$$

To obtain the parametric equations of the different classes of those self-projective quartics which have Stahl sextics, we need only select three linearly

* Stahl, *Journal für Math.*, Vol. CIV, p. 302. This is the unique sextic to which the fundamental involution is apolar, referred to hereafter as the *Stahl sextic*.

† AMERICAN JOURNAL OF MATHEMATICS, Vol. X, p. 70.

‡ Other singular forms are not given, for they furnish no new types. See § 8.

independent second polars of these sextics, written homogeneously. Usually the most convenient are

$$\frac{\partial^2}{\partial t_1^2}, \quad \frac{\partial^2}{\partial t_1 \partial t_2}, \quad \frac{\partial^2}{\partial t_2^2}.$$

Designate henceforth by S_n the sextic that admits a group of order n and by ρ_n^4 the quartic curve derived from it. Note that we get a proper curve for every case except (5).

We proceed now to the consideration of the others in greater detail. In particular, we shall characterize them by means of the fundamental invariants.* We use Professor Morley's set, indicating Salmon's equivalents and the geometric meaning of their vanishing, thus:

Morley:	Salmon:	Condition for:
I_2	$-B$	triple point,
I_2'	$A + 12B$	flex conic to touch,
I_4	R	undulation,
I_6	D	cusp,
I_9	M	self-reflexion.

§ 3. The Cyclic Groups.

Professor Morley has shown that *the necessary and sufficient condition for a ρ^4 to be reflected into itself, i. e., to admit a cyclic g_2 , is*

$$I_9 = 0.†$$

On this supposition simply certain properties of the curve are readily inferred. Thus the center is the intersection of two double lines. The axis cuts out one double point, the intersection of the other double lines and the contacts of tangents from the center, whose parameters are the double points of the involution. *The six points of inflexion are harmonically perspective from the center*, accounting for the two relations that must exist on them.

Consider next the cyclic g_3

$$t' = \omega t, \quad \omega^3 = 1.$$

A sextic to admit this group can contain only cube terms. It is therefore of the form

$$t^6 + \alpha t^3 + \beta.$$

* F. Morley, "Projective Geometry Notes" (1910), Chapter 6. J. E. Rowe, *Trans. Amer. Math. Soc.*, Vol. XII, p. 304. Salmon, "Higher Algebra," 4th ed., §§ 220 ff., where the system of two quartics is to be interpreted as the fundamental involution of the curve.

† *Loc. cit.* Instead of *involutory collineation* we use *reflexion*, the abstract projective notion of physical reflexion.

We can choose the unit point so that $\beta = 1$, reducing the sextic to type (4), and we have the theorem: *A ρ^4 that is invariant under a cyclic g_3 admits at the same time a dihedral g_6 .*^{*}

If the sextic is unaltered by the cyclic g_4

$$t' = it, \quad i^4 = 1,$$

it takes the form

$$t(t^4 + \alpha).$$

The reference scheme may be so selected that $\alpha = 1$, which gives at once the octahedral sextic S_{24} . The ρ^4 derived therefrom has the same property. Therefore:

If a rational quartic is invariant under a cyclic g_4 , it admits also the octahedral g_{24} ; in other words, has three inflexional nodes.

Berzolari† has deduced the properties of this curve from the consideration of the g_{24} of permutations that leaves the double lines unaltered.

Three second polars of the S_5 determine the quartic

$$\begin{aligned} x_0 &= t^4, & x_1 &= t, & x_2 &= 1, \\ x_1^4 - x_0 x_2^3 &= 0. \end{aligned}$$

This curve is unaffected by the one-parameter group of § 1.‡ Hence:

If a ρ^4 admits a cyclic g_5 , it admits an infinite group.

There is an undulation at $t = 0$ and at ∞ the dual singularity, a special triple point whose three parameters have come together, counting for a node and two cusps. All of the point and line singularities are concentrated at these two points.

All of the invariants vanish.§

§ 4. *The Metrical Aspect of Dihedral Collineation Groups.*

We shall now explain a very convenient metrical representation of dihedral collineation groups of any order, in which all of the conjugate points of a set are in general real, and examine the quartics invariant under them. We shall speak in metrical terms at our convenience, leaving the projective interpretation to be supplied if desired.

* The theorems of this section are predicated on the supposition that the Stahl sextic exists and has no repeated roots. Otherwise some of them are not valid.

† "Sulla Lemniscata Proiettiva," *Ist. Lomb. Rend.*, Series II, Vol. XXXVII, pp. 277 and 304.

‡ Cf. Ciani, *loc. cit.*, p. 1175.

§ Cf. the case of the cuspidal cubic.

Using the absolute coordinates

$$\begin{cases} x = X + iY, \\ \bar{x} = X - iY, \end{cases} \quad (X \text{ and } Y \text{ rectangular coordinates}),$$

and an elliptic naming in which the parameter t runs around the unit circle, a point on a rational curve is given by the single equation

$$(1) \quad x = f(t),$$

where f is a rational algebraic function.* This carries with it the conjugate

$$(2) \quad \bar{x} = f(t),$$

derived by writing for each complex number, including t , its conjugate. If x and $\bar{x}$ are interchanged when $t' = \frac{1}{t}$ the curve is symmetrical about the base line, admits a g_2 . If in addition

$$x' = \varepsilon^a x \text{ when } t = \varepsilon t', \quad 1 \leq a \leq n-1,$$

where ε is a primitive n -th root of unity, the curve is invariant under a rotation about the origin of period n , admits a cyclic g_n . The two generate a dihedral g_{2n} containing n rotations, including the identity, and n reflexions.†

Now a g_2 means a line of symmetry, and the cyclic group carries this line into n distinct positions when n is odd. Hence, in that case:

A dihedral g_{2n} means symmetry about n equispaced lines through the origin.

These are axes of the n reflexions, the centers being at infinity.

When n is even there are two sets of $\frac{n}{2}$ lines of symmetry, not conjugate to each other, which make up n equispaced axes of reflexion. Furthermore, the curve has the origin for center.

In either case the centers lie on a line, the line at infinity, and the axes meet at a point, the origin, point and line being fixed elements of the group. When n is even there is a center on each axis.

The elements of the cyclic g_n binary and ternary are

$$(4) \quad t = \varepsilon^a t', \quad \begin{cases} x' = \varepsilon^a x, \\ \bar{x}' = \varepsilon^{-a} \bar{x}, \end{cases} \quad a = 1, \dots, n.$$

* See a memoir by Professor Morley, *Trans. Amer. Math. Soc.*, Vol. I, p. 97.

† Cf. Hilton, "Finite Groups," p. 53, Ex. 21 (iii).

The products of these with the g_2 's

$$(5) \quad t = \frac{1}{t'}, \quad \begin{cases} x' = \bar{x}, \\ \bar{x}' = x, \end{cases}$$

are the n involutions and reflexions respectively.

The application of the ternary g_n to a point and its image affords an easy geometrical construction for the general set of $2n$ conjugate points. Such sets break up into two concentric regular polygons of n vertices. If, however, a point is at a center or lies on an axis of reflexion, it assumes only n distinct positions.

Among these special sets of n points on ρ^m are the two sets whose parameters are the fixed points of the involutions and which may be taken as the basis of determination of the conjugate sets of the binary group. In the canonical form employed here, the general set of $2n$ conjugate points of the binary group is given by

$$(6) \quad (t^n + 1)^2 - \lambda (t^n - 1)^2,$$

and therefore factors into two sets of n .*

Among the curves left invariant by the general dihedral group is the system of concentric circles (double contact conics)

$$(7) \quad x\bar{x} = \lambda^2.$$

These will intersect ρ^m in $2m$ invariant points which break up into conjugate sets, their parameters being conjugate sets of the binary group.

This furnishes a method not only for a geometrical construction of conjugate sets of the ternary group, but also for finding the analytic representation of such sets of the binary group.

§ 5. *The Dihedral ρ_4^4 .*

From the typical sextics we get immediately the equations of the dihedral quartic curves

$$(1) \quad x = \frac{\frac{\partial^2 S}{\partial t_2^2}}{\frac{\partial^2 S}{\partial t_1 \partial t_2}}, \quad \bar{x} = \frac{\frac{\partial^2 S}{\partial t_1^2}}{\frac{\partial^2 S}{\partial t_1 \partial t_2}}.$$

We may write the quartic with a dihedral g_4 ,

$$(2) \quad x = \frac{t^3}{t^4 + at^2 + 1}, \quad \bar{x} = \frac{t}{t^4 + at^2 + 1}.$$

* Klein, *loc. cit.*, p. 49.

Since $n = 2$, the curve is symmetrical about each of two perpendicular lines and their point of intersection. A general set of four conjugate points therefore will form a rectangle with center at the origin.

The three elements of this group, omitting the identity, are involutory, since the rotation is of period 2 and therefore a reflexion from the origin in the line at infinity.

Now the parametric equations show that there is a bi-flecnode, isolated however, at the origin. Hence, if a ρ^4 admits a dihedral g_4 , it has a bi-flecnode. Conversely, if a ρ^4 has a bi-flecnode, it is invariant under a dihedral g_4 , for its equation can be written

$$(3) \quad y^2 z^2 + z^2 x^2 + x^2 y^2 + 2 f x^2 y z = 0,$$

which is unaltered by changing the sign of x , interchanging y and z , or changing the signs of y and z interchanged; i. e., by a dihedral g_4 .

Here, then, is an example of a rational quartic that is reflected into itself from a point on it. This transformation is singled out from the other two which are symmetrical in their effects.

Denote the three reflexions by R_1, R_2, R_3 , their centers by a_1, a_2, a_3 , and the axes by $\alpha_1, \alpha_2, \alpha_3$; a_3 being the bi-flecnode and a_2 on the base line.

Two of the double lines meet at a_1 and the other two at a_2 . Again, the other four flexes lie on three pairs of lines, one pair on each center, and the flex tangents meet in three pairs of points, one pair on each axis. Or we may say:

The diagonal 3-point of the four inflexional points and the diagonal 3-line of the four inflexional lines constitute a single triangle, the centers and axes of reflexion.

This is a fixed triangle of the group and characterizes it geometrically.

The fundamental involution is

$$(a t^4 - 6 t^2) + \lambda (6 t^2 - a),$$

and the invariants are

$$\begin{aligned} I_2 &= -a^2, & I'_2 &= -a^4, & I_4 &= a^4 (a^2 - 36)^2, \\ I_6 &= -16 a^6, & I_9 &= 0, & I'_6 &= 64 a^6 (a^2 - 4), \end{aligned}$$

whence the invariant conditions on the curve are

$$(4) \quad 16 I_2^3 - I_6 = 0, \quad I_4 - (I'_2 - 36 I_2)^2 = 0.$$

The second of these is the condition for four collinear flexes, which we saw is twice satisfied.

Since an absolute constant remains, the curve may be further specialized. A cusp can not occur, for that would imply too many intersections with the axis α_3 . *The two double points may come together, however, forming a tac-node at either a_1 or a_2 with α_3 as the nodal tangent, the condition being $a^2 = 4$, the I'_6 above. We have then a second center of reflexion lying on the curve; viz., at the tac-node.*

Equations (2) become

$$(5) \quad x = \frac{t^3}{(t^2 \pm 1)^2}, \quad \bar{x} = \frac{t}{(t^2 \pm 1)^2},$$

which give, on eliminating t ,

$$(6) \quad (x \pm \bar{x})^4 - x\bar{x} = 0.$$

If $a^2 = 36$, $I_4 = 0$ and the curve acquires two undulations, given by $t^2 \mp 1$ and lying on α_1 or α_2 , corresponding to $\pm a$. Thus the two undulations, the inflexional node and the intersection of the two double lines are collinear.

The single condition $a = 0$ suffices for two additional bi-flecnodes, and the curve becomes a lemniscate.

§ 6. *The Dihedral ρ_6^4 .*

Taking for the S_6

$$t^6 + 5at^3 + 1,$$

we obtain the equations of ρ_6^4 in the canonical form

$$(1) \quad x = \frac{at^3 + 1}{t^2}, \quad \bar{x} = \frac{t^4 + at}{t^2}.$$

Here n is odd and equal to 3, so that the curve has symmetry about three equispaced lines through the origin. On each of these axes of reflexion lies a double point, while all three centers are on a double line.

*The group is characterized geometrically as the g_6 that permutes the sides of the triangle of double points.**

All of the single conics, of which there is a multitude, attached to the ρ^4 , belong to the system of concentric circles

$$x\bar{x} = \lambda^2.$$

* Brusotti, *Ist. Lomb. Rend.* (II), Vol. XXXVII, p. 888, discusses a class of rational curves that admit this group, namely those with three *hyper-osculation* points, undulations for the ρ^4 .

To the different values of a correspond projectively distinct ρ_6^4 's. The line at infinity is, however, a common double line having contacts with each at the circular points.

The fundamental involution is

$$(4t^3 + a) + \lambda(a t^4 + 4t).$$

The values of the invariants and the geometrical significance of their vanishing are

$$\begin{array}{ll} I_2 = a^2 - 1, & \text{triple point,} \\ I_2' = -(a^2 + 2)^2, & \text{3 bi-flecnodes,} \\ I_4 = a^2(a^2 - 16)^2, & \text{3 undulations,} \\ I_6 = -a^2(a^2 - 4)^3, & \text{3 cusps.} \end{array}$$

Thus, on account of the triangular symmetry, if a condition is satisfied once, it is, generally speaking, satisfied three times.

The invariant conditions on ρ_6^4 are

$$\begin{aligned} I_9 &= 0, \\ [I_4 + (4I_2 - I_2')(60I_2 + I_2')]^2 - 768I_6'(12I_2 + I_2') &= 0, \end{aligned}$$

where I_6' is the tac-node condition.

The equation in absolute coordinates of the curve with a triple point ($a = 1$) is

$$x^3 + \bar{x}^3 - x^2\bar{x}^2 = 0.$$

The sixteen intersections with the deltoid ($a = 2$), in terms of the parameter of the latter, are given by

$$t^2(t^6 + t^3 + 1)^2.$$

Hence, the two curves have eight contacts at which the eighteen common lines are accounted for, the common double tangent counting for six.

We note that, if a ρ^4 have three cusps or three undulations, it must admit a dihedral g_6 , since it is then determined by a binary cubic.*

§ 7. The Projective Lemniscate with a Real Dihedral g_8 .

The octahedral group contains as a sub-group a dihedral g_8 . We include a brief mention of the ρ_{24}^4 with reference to the g_8 , since in this scheme of representation the sets of eight conjugate points are in general real.

* Cf. S_6 with three double roots, below.

The metrical equations, found at once from S_{24} , are

$$x = \frac{t}{t^4 + 1}, \quad \bar{x} = \frac{t^3}{t^4 + 1},$$

whence

$$(x^2 + \bar{x}^2)^2 - x\bar{x} = 0.$$

The curve is symmetrical about each of two pairs of perpendicular lines which make up a set of four equispaced lines through the origin, which is an isolated node.

The fundamental involution is

$$a(t^4 + 1) + \lambda b t^2,$$

a quartic and its Hessian.

The invariant relations are

$$I'_2 = 0, \quad I_9 = 0,$$

$$I_4 - 1296 I_2^2 = 0,$$

$$I_2 I_4 - 81 I_6 = 0.$$

The curve cuts the base circle at the twelfth roots of unity, exclusive of the fourth roots. Bearing in mind the symmetry it is readily drawn.*

§ 8. *Singular Stahl Sextics.*

The table of Bolza gives only non-singular sextics. We shall now examine those with multiple roots. A triple root can not occur, for the second polars all have a common factor, the ∞^2 line sections reducing to a pencil, and the curve is degenerate. There may be, however, one, two or three double roots.

Starting with Bolza's types, we may impose the further conditions for one or more repeated factors, obtaining thus singular S_i 's. But if we first insist on repeated roots and then ask that the sextics admit the different groups, will we get the same tabulation? In other words, are the processes commutative?

Beginning with the extreme case, suppose the sextic has three double roots. It is then the square of a cubic and may be taken as

$$(t^3 + 1)^2.$$

Hence it is a special case of S_6 , $\alpha = 2$. We saw that the three axes of reflexion contain each a double point of ρ_6^4 . In this case the other intersections of the axes are given by

$$t^3 + 1 \quad \text{and} \quad t^3 - 1.$$

* This is the *right circular cross curve* of Loria, "Spezielle Ebene Kurven," Vol. I, p. 226.

Starting with $(t^3 - 1)^2$, we get the same curve, since the two sextics are projectively equivalent.

A sextic with two double roots is made up of one quadratic and the square of another, and may be taken as

$$t(t^2 + \alpha t + 1)^2.$$

Since there is always an involution leaving two quadratics unaltered; *a sextic with two double roots admits a g_2* . This is a special case of S_2 .^{*} If we ask further that the sextic admit the involution $t' = -t$, the double roots must be interchanged and it may be reduced to the form

$$t(t^2 \pm 1)^2,$$

and is a special case of type (3), $\alpha = \pm 2$. In this case the double points of the three involutions of the g_4 are the two quadratic factors of the sextic and their Jacobian, three mutually apolar pairs. *The corresponding ρ^4 is an instance of one whose Stahl sextic is cut out by a line, the axis α_1 or α_2 .*

The sextic with one double root may be written

$$t^2(t^4 + at^3 + bt^2 + ct + 1).$$

Now the double root 0 must be a fixed point in any transformation that leaves the sextic unaltered, while all the transformations that leave one point unaltered constitute a cyclic group that leaves a second point unaltered. Two cases are to be distinguished.

CASE I. *The second fixed point not a root of S .* Then the involution $t' = -t$ will interchange the four simple roots in pairs and S becomes

$$t^2(t^4 + at^2 + 1),$$

which is a special case of type (1), as is readily verified. S can not admit a cyclic g_3 , for one of the simple roots would be a fixed point contrary to hypothesis. The four points, however, might be permuted cyclicly by the transformation

$$t' = it, \quad i^4 = 1,$$

when S assumes the form

$$t^2(t^4 + 1).$$

The ρ^4 corresponding to this may be written

$$\begin{aligned} x_0 = t, \quad x_1 = t^2, \quad x_2 = t^4 + 1, \\ x_0^4 + x_1^4 - x_0^2 x_1 x_2 = 0. \end{aligned}$$

It has a triple point with a cusp at $0, \infty$.

^{*} Writing S_2 , $a t_1^6 + b t_1^5 t_2 + t_1^4 t_2^2 + c t_1^3 t_2^3 + t_1^2 t_2^4 + b t_1 t_2^5 + a t_2^6$, we obtain, on placing $a = b = 0$, $t_1^2 t_2^2 (t_1^2 + c t_1 t_2 + t_2^2)$, a form to which a sextic with two double roots may be reduced.

CASE II. *The second fixed point a root of S .* The sextic can now admit the cyclic g_3

$$t' = \omega t, \quad \omega^3 = 1,$$

which permutes cyclicly the remaining roots producing

$$S_3, \quad t^2 (t^3 + 1).$$

The corresponding ρ^4 may be written

$$x_0 = 4t^3 + 1, \quad x_1 = t^4 + 4t, \quad x_2 = t^2,$$

which is a curve with three undulations, and is therefore a special case of the ρ_6^4 already noticed, § 6.*

There are then sextics with one repeated root invariant under cyclic groups of orders 2, 3 and 4, only the last of which, however, determines a new type of curve.

§ 9. *Cases of Failure.*

There remains the consideration of the ρ^4 with a singularity such that the tangent cuts out a quartic with four equal roots, for then the fundamental involution has a common factor and the Stahl sextic is no longer defined.

If ρ^4 have an undulation, say at the point 0, it can admit only a cyclic group of which the undulation is a fixed point. The group will have another fixed point, say ∞ . Referred to the tangents at these two points and the line joining them, the equations of the curve are

$$\begin{cases} x_0 = t^4, \\ x_1 = t^2 + b t + 1, \\ x_2 = t (t^2 + c t + a). \end{cases}$$

This will admit the involution

$$t' = -t, \quad \text{if } b = c = 0,$$

and we have as the equations of a ρ_2^4 with an undulation

$$x_0 = t^4, \quad x_1 = t^2 + 1, \quad x_2 = t (t^2 + a),$$

which is a special case of ρ_2^4 of § 3.

If the other fixed point is an inflexion, ρ^4 may admit the cyclic g_3 , $t' = \omega t$. The equations are

$$x_0 = t^4, \quad x_1 = t, \quad x_2 = t^3 + 1,$$

or

$$x_1 (x_0 + x_1)^3 - x_0 x_2^3 = 0.$$

* It should be remarked that while the sextic admits only a g_3 there is no contradiction in our theory, for the Stahl sextic in this case is not unique (see below).

The flex tangent x_1 meets again at the undulation and x_2 cuts out a triple point. Three flexes and three double lines remain.

Again, the quartic with a tac-node-cusp at 0 may admit the cyclic g_3 , giving rise to a new type of ρ^4_3 ,

$$x_0 = t^4, \quad x_1 = t^2, \quad x_2 = t^3 + 1,$$

or

$$x_1 (x_0^3 - x_1^3 + 2 x_0 x_1 x_2) - x_0^2 x_2^2 = 0.$$

The multiple point consumes all of the singularities of the curve except three flexes.

A ρ^4 with two undulations, say at 0 and ∞ , may be written

$$x_0 = t^4, \quad x_1 = 1, \quad x_2 = t (t^2 + 2 a t + 1),$$

which admits the involution

$$t' = 1/t.$$

Hence, a ρ^4 with two undulations is invariant under a g_2 . In this case the undulations are interchanged. They may be fixed points of an involution when the curve takes the form

$$x_0 = t^4, \quad x_1 = 1, \quad x_2 = t (t^2 + 1),$$

which is invariant under a dihedral g_4 and is the same as that noticed, § 5.

The ρ^4 with three undulations is Brusotti's curve and admits a dihedral g_6 , § 6, foot-note. It may be derived uniquely from the special S_6

$$t^6 + 20 t^3 + 1.$$

But if we attempt to recover the sextic, we obtain the doubly infinite system

$$(t^6 + 20 t^3 + 1) + \lambda t^2 (2 t^3 + 5) + \mu t (5 t^3 + 2),$$

any member of which determines the same curve.

We have thus arrived at a complete solution of the first part of our fundamental problem, having found eight types of self-projective rational quartics. Moreover, the nature of the investigation is such that we are in a position to say that *these eight types include as special cases all others*.

For convenience of reference they have been collected in a table at the end.

II. THE RATIONAL QUINTIC.

§ 10. General Considerations.

We begin our study of the ρ^5 with a generalization of a theorem enunciated for the ρ^4 , § 3, viz.:

If a rational curve ρ^m of order m

$$x_i = (\alpha_i t)^m, \quad i = 0, 1, 2,$$

is invariant under a cyclic group of order n , $n > m$, it is invariant under a one-parameter group.

Proof: A ternary collineation of period n can always be written in the canonical form

$$x'_i = \varepsilon_i x_i,$$

where the ε 's are n -th roots of unity. The triangle of reference is then a fixed triangle of the group.* Now each side of this triangle must cut ρ^m in m points whose parameters form a conjugate set under the corresponding binary cyclic g_n . If $n > m$, the only conjugate sets of m points are

$$t_1^a t_2^b, \quad a + b = m.$$

The only three sets having no common factors are

$$t_1^m, \quad t_2^m, \quad t_1^{m-r} t_2^r, \quad 1 \leq r \leq m-1.$$

The equations of the curve then become

$$x_0 = t_1^m, \quad x_1 = t_1^{m-r} t_2^r, \quad x_2 = t_2^m,$$

which admits an infinite group.

If $n = m$, the only conjugate sets of n points are

$$t_1^a t_2^b \quad \text{and} \quad \alpha t_1^n + \beta t_2^n, \quad a + b = n.$$

There can not be three independent sets of the latter kind. If there are two independent sets, they can be replaced linearly by t_1^n or t_2^n , and we have the one-parameter type. If there is one such set and a set t_1^n or t_2^n , we can obtain by a transformation the preceding case. The only new types then are

$$\alpha t_1^n + \beta t_2^n, \quad t_1^r t_2^{n-r}, \quad t_1^s t_2^{n-s}, \quad r \neq s, \quad 1 \leq r, s \leq n-1.$$

If there is no set of the form $\alpha t_1^n + \beta t_2^n$, we have again the one-parameter case.

That is, the equations of a ρ^n invariant under a cyclic g_n may be written (non-homogeneously) in canonical form

$$x_0 = t^r, \quad x_1 = t^s, \quad x_2 = t^n + 1,$$

with the above restrictions imposed on r and s .

* Every such collineation has at least one proper fixed triangle, an homology having ∞^1 .

These equations are the canonical form of ρ^n invariant under a dihedral g_{2n} if $r + s = n$.

Incidentally, for varying r in the one-parameter case, we obtain the classes of curves invariant under an infinite group. These are the self-dual binomial curves, so-called parabolas of higher order,

$$x_1^n - x_0^{n-r} x_2^r = 0, \quad 0 < r < n.$$

The singularities are all concentrated at the two fixed points 0 and ∞ . There are two such types of ρ^5 , $r = 1, 2$.^{*} If $r = 3, 4$, we obtain curves metrically distinct but projectively equivalent.[†]

Our theorem excludes for the rational quintic those cyclic g_n 's and dihedral g_{2n} 's, $n > 5$. Furthermore, a group that transforms a ρ^5 into itself must leave the flex nix unaltered. From a study of the characteristic invariant forms of the binary group — i. e., those special sets of parameters which assume fewer than n values — we readily conclude that

A ρ^5 can not admit a dihedral g_{2n} , n even, nor the tetrahedral, the octahedral or the icosahedral group, even though its flex equation have multiple roots.

There are left, then, only cyclic groups of orders 2, 3, 4 and 5 and dihedral groups of orders 6 and 10. It turns out that there are rational quintics belonging to all of these.

§ 11. The Cyclic Groups.

Using the fixed triangle of the group as triangle of reference, we obtain without difficulty the equations of the cyclic quintics in canonical form.

Thus the ρ_2^5 is

$$\begin{cases} x_0 = t^5 + a t^3 + b t, \\ x_1 = t^4 + c t^2, \\ x_2 = t^2 + 1. \end{cases}$$

The only fixed points under the involution are 0 and ∞ . One of these is an inflexion — the center of the reflexion. The other lies on the axis, being the contact of a tangent from the center. The parameters of the other contacts of tangents from the center and those of the extra intersections of the axis must be interchanged by the involution. That is, *two double lines meet at the center and two double points lie on the axis.*

^{*} Cf. Snyder's G_{20} and G_{30} , *loc. cit.*, paragraphs 5 and 6. It may be remarked that his curves $axy^4 + by^5 + cz^5 = 0$ and $axy^4 + bz^5 = 0$ are projectively equivalent.

[†] See Wieleitner, "Algebraische Kurven," p. 135, Beisp. 5.

The equations of ρ_3^5 are

$$\begin{cases} x_0 = t^5 + a t^2, \\ x_1 = b t^3 + 1, \\ x_2 = t^4 + t. \end{cases}$$

The points of the curve are arranged in triads, but as the singularities occur in multiples of 3 this can happen without any striking geometrical specialization. The line x_2 is the apolar line and cuts out the fixed points of the generating binary transformation.

We write the ρ_4^5

$$\begin{cases} x_0 = a t^5 + t, \\ x_1 = t^3, \\ x_2 = t^4 + 1. \end{cases}$$

Now the binary cyclic g_4 generated by

$$t' = i t, \quad i^4 = 1,$$

contains as a subgroup the involution $t' = -t$. Hence the ρ_4^5 admits a reflexion into itself. The line x_1 is seen to be a touching flex tangent, the contacts being the fixed points of the binary group. The line x_2 is the axis of reflexion and cuts out as above two double points. In addition to the special flex tangent two double lines meet at 0. Thus, *the center of reflexion is a syzygetic point*.*

For particular values of r and s of the previous section we obtain two types of ρ^5 invariant under the g_5 generated by

$$t' = \varepsilon t, \quad \varepsilon^5 = 1,$$

$$(1) \quad x_0 = t, \quad x_1 = t^2, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 + x_1^5 - x_0^3 x_1 x_2 = 0; \dagger$$

$$(2) \quad x_0 = t^3, \quad x_1 = t, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 - x_1^5 + x_0 x_1^2 x_2^2 - 2 x_0^3 x_1 x_2 = 0.$$

The first has a fourfold point at $0, \infty$, with three coincident parameters, consuming four flexes and seven double lines. The parameters are the fixed points of the binary group.

* By a *syzygetic point* we mean a point whose lines cut out the syzygetic pencil of binary quartics $U + \lambda H$, where H is the Hessian of U . See *Johns Hopkins University Circular*, February, 1911, p. 101.

Snyder's curve, paragraph 5, is a special case of this and may be written parametrically

$$x_0 = t^5 + t, \quad x_1 = t^4, \quad x_2 = 1,$$

when 0 names the undulation, ∞ the hyper-osculation point and $t^4 + 1$ the fourfold point.

† Cf. Snyder, *loc. cit.*, paragraph 5.

The second has a cusp of higher order at a point of inflexion which uses up all of the double points, four flexes and seven double lines, the parameters being fixed under the binary group.

Sets of five conjugate points on either curve are cut out by the pencil of double contact conics

$$x_0^2 + \lambda x_1 x_2 = 0,$$

five of the intersections being taken up at the multiple point.

§ 12. The Dihedral ρ_{10}^5 's.

We find likewise two types of quintics invariant under the dihedral g_{10} obtained by combining with the g_5 above the involution

$$t' = 1/t.$$

They are

$$(1) \quad x_0 = t^3, \quad x_1 = t^2, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 + x_1^5 - x_0^2 x_1^2 x_2 = 0; *$$

$$(2) \quad x_0 = t, \quad x_1 = t^4, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 + x_1^5 + x_0 x_1 x_2 (3 x_0 x_1 - x_2^2) = 0.$$

The first has a fourfold point with two cusps, the parameters being fixed under the cyclic g_5 . The line x_2 cuts out the five flexes which are the centers of the five reflexions. From each inflexion run one of the five double lines and one simple tangent. The contacts of the latter, together with the corresponding flexes, make up the five pairs of fixed points of the involutions.

In the metrical representation of § 4 the curve has fivefold symmetry. The equations are †

$$x = \frac{t^3}{t^5 + 1}, \quad \bar{x} = \frac{t^2}{t^5 + 1}.$$

The multiple point is isolated at the origin but the five double lines are real, their equations being

$$5 (\epsilon^i x + \epsilon^{-i} \bar{x}) + 4 = 0, \quad i = 0, \dots, 4; \epsilon^5 = 1.$$

There are five inflexional asymptotes whose equations are

$$5 (\epsilon^i x + \epsilon^{-i} \bar{x}) - 1 = 0.$$

* This is Snyder's curve (2), paragraph 5.

† Cf. Hulburt, "Calculus," Ex. 14, p. 125.

We saw that sets of conjugate points were cut out by the system of concentric circles

$$x \bar{x} = \lambda^2.$$

Among these may be mentioned the circle on

- | | | |
|-----|------------------------------|------------------|
| (a) | 5 flex lines, | $10\lambda = 1,$ |
| (b) | 5 lines from flexes, | $2\lambda = 1,$ |
| (c) | 5 double lines, | $5\lambda = 2,$ |
| (d) | 10 contacts of double lines. | |

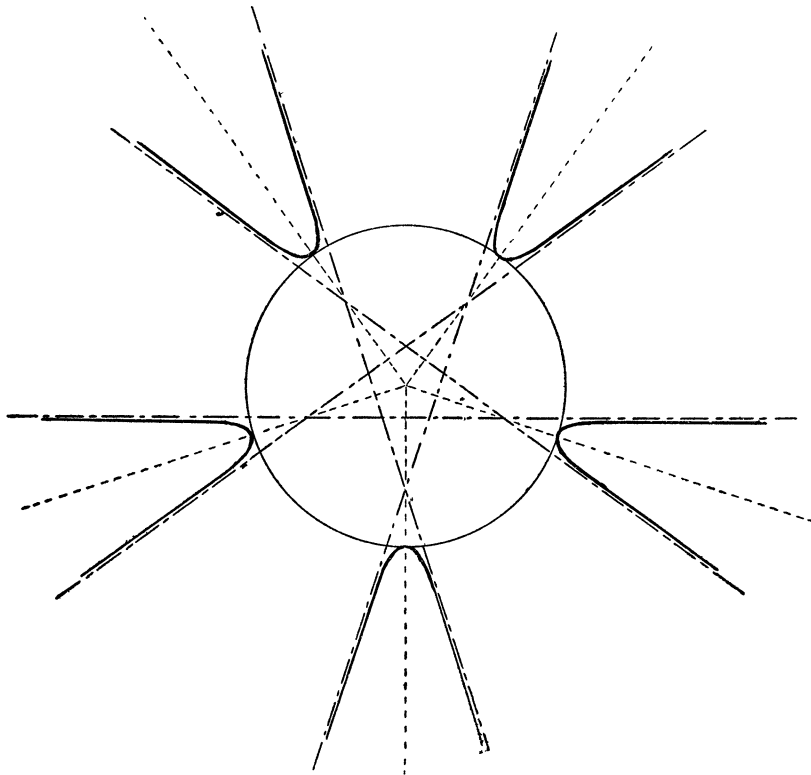


FIG. 1.

(b) has five contacts with ρ^5 , lying on the axis and given by $t^5 - 1$.

The curve cuts the base circle at the fifteenth roots of -1 , exclusive of the fifth roots. A figure (Fig. 1) has been drawn, using the equation $x' = 5x$.

Quintic (2) has a double point of which each branch is an undulation,* the nodal parameters being the fixed points of the binary cyclic g_5 . The line x_2

*This may be called a *bi-und-node*.

cuts out the remaining five flexes which are the centers of the five reflexions. From each flex run one ordinary tangent and two of the ten double lines.

Metrically the equation of the curve is

$$x = \frac{t}{t^5 + 1},$$

The bi-und-node, at which the five axes of reflexion meet, is isolated at the origin. The other five double points are real, one lying on each axis and together forming a special conjugate set of the ternary group and their parameters a general set of the binary group. That is,

The six double points lie at the vertices and center of a regular pentagon. They are therefore perspective in ten ways, two ways corresponding to each axis of symmetry.

There are five real inflexional asymptotes:

$$5 (\epsilon^i x + \epsilon^{-i} \bar{x}) + 3 = 0.$$

Among the system of invariant circles we note that on

- | | | |
|-----|----------------------|-------------------|
| (a) | 5 flex lines, | $10 \lambda = 3,$ |
| (b) | 5 lines from flexes, | $2 \lambda = 1,$ |
| (c) | 5 double points, | $\lambda = 1,$ |

besides two each on

- | | |
|-----|------------------------------|
| (d) | 5 double lines, |
| (e) | 10 contacts of double lines. |

(b) is the perspective conic, or we may say, *the perspective conic is inscribed in the five tangents from the flexes, the contacts lying on the axes of symmetry, given by $t^5 - 1$.* These contacts and the flex parameters, therefore, are the fixed points of the involutions.

A figure (Fig. 2) is given.

§ 13. The Dihedral ρ_6^5 .

The equations of ρ^5 invariant under the dihedral g_6 generated by

$$t' = \omega t, \quad t' = 1/t, \quad \omega^3 = 1.$$

are found at once from those of ρ_3^5 by writing $a = b$:

$$x_0 = t^5 + a t^2, \quad x_1 = a t^3 + 1, \quad x_2 = t^4 + t.$$

The centers of reflexion are points of inflexion and are given by $t^3 + 1$, That is, *the quadratic giving the fixed points of the binary cyclic g_3 is the*

Hessian of the cubic which names the centers of reflexion, and both are cut out by the apolar line. Denote the cubic by $C_{1,3}$, the cubicovariant by $C_{3,3}$ and the Hessian by H .

From each center run one tangent and two double lines, the contacts $C_{3,3}$ of the former lying on the axes. The residual intersections of each axis, therefore, are made up of two double points. Hence, *the double points are perspective.*

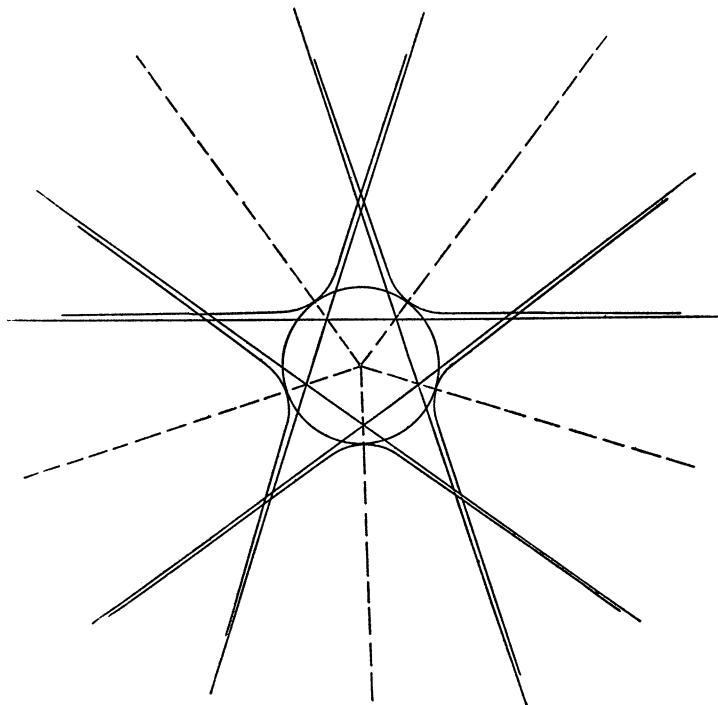


FIG. 2.

The tangents at the centers are

$$3(\omega^i x_0 + \omega^{-i} x_1) + (a+5)x_2 = 0, \quad i = 0, 1, 2.$$

The invariant conics all touch the curve at the Hessian points. Among them is the perspective conic

$$\xi_0 = 1, \quad \xi_1 = t^2, \quad \xi_2 = -(a+1)t,$$

or

$$(a+1)^2 \xi_0 \xi_1 - \xi_2^2 = 0;$$

$$x_0 = (a+1)t^2, \quad x_1 = (a+1), \quad x_2 = 2t,$$

or

$$4x_0 x_1 - (a+1)^2 x_2^2 = 0,$$

which touches again at the cubicovariant points. Hence, *the perspective conic*

is inscribed in the three tangents from the centers of reflexion and touches the curve at the same points.

The group may be characterized geometrically as the one that permutes the sides of this triangle.

The flex form, exclusive of $t^3 + 1$, is

$$a t^6 + (a^2 - 9 a + 10) t^3 + a,$$

and names a general set of six conjugate points. Owing to the symmetry, the vanishing of the discriminant

$$(a - 1) (a - 2) (a - 5) (a - 10)$$

means three equal roots and is therefore the condition that they reduce to a special set of three.

Many interesting special cases of ρ_6^5 arise for numerical values of a . In particular are those corresponding to the roots of the discriminant above. We note the following:

(1)	$a = 1$,	degenerate,	
(2)	$a = 2$,	undulations	at $C_{3,3}$,
(3)	$a = 5$,	cusps	" $C_{3,3}$,
(4)	$a = 10$,	hyper-osculation points	" $C_{1,3}$,
(5)	$a = -2$,	syzygetic points	" $C_{1,3}$,
(6)	$a = -8$,	syzygetic points	" H .

(3) is self-dual. The cuspidal tangents are the axes and therefore meet at a point.

(4) is the quintic case of Brusotti's curves. From each hyper-osculation point run one tangent and one of the three double lines, contacts of the former being $t^3 - 1$. Many of the facts brought out by him in connection with the group apply to all quintics invariant under it.

§ 14. *Three Syzygetic Points.*

I shall conclude the discussion of the quintic with an account of the special ρ_6^5 which suggested this investigation.

Cayley* proved that the locus of the foci of parabolas on three points is a rational quintic passing through the circular points. Now at each of these points three double lines meet,† *i. e.*, they are syzygetic points.

* "Collected Papers," Vol. VII, p. 568.

† Loria, "Ebene Kurven," Vol. I, p. 239 a.

In absolute coordinates the equation can be written *

$$(1) \quad x = t + \sum_1^3 \frac{A_i}{t - \alpha_i},$$

where α_i are the parameters of the three points on the unit circle and

$$A_i = (\alpha_i - \alpha_j)(\alpha_i - \alpha_k), \quad i \neq j \neq k.$$

In this form the coefficients of (1) can be expressed in symmetric functions of the α 's.

The question naturally arises as to the nature of the locus when a particular arrangement of the three base points is selected. For example, if the triangle is equilateral, say $-1, -\omega, -\omega^2$, so that $s_1 = s_2 = 0, s_3 = -1$, equation (1) becomes

$$(2) \quad x = \frac{t^4 - 8t}{t^3 + 1},$$

which is the metrical representation of the sixth type of the previous section. Hence, *the locus of the foci of parabolas on the vertices of an equilateral triangle is a special ρ_6^5* , as is evident also from other considerations. All twelve double lines cut ρ^5 at points of x_2 , two each at the inflexions $t^3 + 1$ and three each at the Hessian points.

Again, if the vertices of the triangle are $-\omega, -\omega^2, 1$, whence $s_1 = s_2 = 2s_3 = 2$, the equations of the curve written homogeneously are

$$(3) \quad \begin{cases} x_0 = t^5 - 2t^4 + 4t^2 - 2t, \\ x_1 = 2t^4 - 4t^3 + 2t - 1, \\ x_2 = t^4 - 2t^3 + 2t^2 - t. \end{cases}$$

It is readily verified that *there is a third syzygetic point, $t = 1$, lying on a line with the other two.*

For the sake of symmetry we apply to the parameter the transformation

$$T: \quad t' = \frac{\omega t + 1}{t + \omega}, \quad T^{-1}: \quad t = \frac{1 - \omega t'}{t' - \omega},$$

so that the syzygetic points are named by $t^3 - 1$. Call this cubic $\bar{C}_{1,3}$, the cubcovariant $\bar{C}_{3,3}$ and the Hessian H . Likewise transfer to a new triangle of reference by the substitution

* *Johns Hopkins University Circular*, February, 1911, p. 105. The proof is given in the manuscript of my dissertation, *Johns Hopkins University*, 1912. The parameters of the absolute points on circle and quintic are 0 and ∞ , while the α 's on the quintic are cut out by the line joining them.

$$X: \begin{cases} x'_0 = x_0 - \omega x_1 + (\omega - 1) x_2, \\ x'_1 = -\omega x_0 + x_1 + (\omega - 1) x_2, \\ x'_2 = (\omega - 1) x_2, \end{cases} \quad X^{-1}: \begin{cases} x_0 = x'_0 + \omega x'_1 + \omega^2 x'_2, \\ x_1 = \omega x'_0 + x'_1 + \omega^2 x'_2, \\ x_2 = \omega^2 x'_2. \end{cases}$$

Under the transformations T and X equations (3) reduce, on dropping primes,* to

$$(4) \quad x_0 = t^5 + 2t^2, \quad x_1 = -2t^3 - 1, \quad x_2 = t^4 - t,$$

or, eliminating t ,

$$(5) \quad x_0 x_1 (x_0^3 + 3x_0 x_1 x_2 + x_1^3) + x_2^2 (2x_0^3 + x_0 x_1 x_2 + 2x_1^3) - x_2^5 = 0.$$

We saw, § 13, that the syzygetic points are inflexions. It is readily found that *the tangents at the syzygetic points touch again at $\bar{C}_{3,3}$* . These three pairs of parameters, then, are the fixed parameters of the involutions. Now each such touching flex tangent is made up of one stationary line and two double lines. *The tangents from the syzygetic points therefore account for the twelve double lines of the curve.*

The equations of the perspective conic are

$$(6) \quad \begin{cases} \xi_0 = 1, & \xi_1 = t^2, & \xi_2 = t, \\ x_2^2 - 4x_0 x_1 = 0. \end{cases}$$

Substituting the values of the x 's from (4), the contacts are found to be

$$t(t^3 + 1).$$

Hence, *the perspective conic is inscribed in and has contacts in common with the five tangents at the points $H \bar{C}_{1,3}$, cut out by the apolar line, the contacts being given by $H \bar{C}_{3,3}$* . Since each syzygetic tangent counts for four common lines of the two curves, these five defining tangents constitute the total of sixteen.

We proceed in the next instance to a consideration of the double points. The coordinates of a node of (1) were found to be †

$$\begin{cases} x_0 = s_3 \alpha_3 (\alpha_2 + \alpha_3) - \alpha_2 \alpha_3^3 (\alpha_2 - \alpha_3) - s_3 \alpha_1 \alpha_2, \\ x_1 = \alpha_3^2 (\alpha_2 - \alpha_3) - \alpha_2^2 (\alpha_3 - \alpha_1) + s_3, \\ x_2 = s_3 \alpha_3, \end{cases}$$

* On writing $t = -t$ and changing the signs of x_0 and x_1 , this becomes the special case (5) of ρ_6^5 above.

† *Johns Hopkins University Circular, loc. cit.*

whence the others can be derived by permutations of the α 's. For this case they become, substituting the particular values of the α 's,

$$\begin{array}{ccc} 1 - \omega^2, & 1 - \omega^2, & \omega^2 - \omega, \\ 1 - \omega^2, & \omega^2 - \omega, & 1 - \omega^2, \\ 1, & 1, & 1; \\ 1 - \omega, & \omega - \omega^2, & 1 - \omega, \\ 1 - \omega, & 1 - \omega, & \omega - \omega^2, \\ 1, & 1, & 1. \end{array}$$

Under the transformation X these reduce to

$$\begin{array}{ccc} (1, 1, \omega), & (1, \omega, 1), & (\omega, 1, 1); \\ (1, 1, \omega^2), & (1, \omega^2, 1), & (\omega^2, 1, 1). \end{array}$$

Denote the first row by d_1, d_2, d_3 and the second by d'_1, d'_2, d'_3 .

These two triangles of double points, together with

$$\begin{array}{ccc} (1, 0, 0), & (0, 1, 0), & (0, 0, 1); \\ (1, 1, 1), & (1, \omega, \omega^2), & (1, \omega^2, \omega), \end{array}$$

have the canonical form of four triangles, any two of which are in sixfold perspection, the centers of perspection being the other two. Hence, *the six double points form two fully perspective triangles. In other words, they define a Hessian configuration in the plane, namely, that associated with a syzygetic pencil of cubic curves.** This just accounts for the four relations that must exist among them.

The little table exhibits in rows and columns the six centers of perspection, together with the lines passing through them.

$d_1 d'_1$	$d_2 d'_3$	$d'_2 d_3$	0, 0, 1
$d_2 d'_2$	$d_3 d'_1$	$d'_3 d_1$	0, 1, 0
$d_3 d'_3$	$d_1 d'_2$	$d'_1 d_2$	1, 0, 0
1, 1, 1	1, ω , ω^2	1, ω^2 , ω	

The residual intersections of the fifteen lines joining the double points are given by $H^3 \bar{C}_{1,3}^2 \bar{C}_{3,3}$. Hence, two of the centers are on the curve, *viz.*, at the Hessian points.

* Three of the base points of the pencil of cubics are the syzygetic points of the quintic.

Since the curve admits three reflexions, the other six inflexions must lie in harmonic pairs on three lines from each of the syzygetic points. That is, *they are in triple harmonic perspection, which just accounts for the four conditions on them.*

Thus, from each syzygetic point run, besides the touching flex tangent, two double lines and two lines carrying each two double points, and three lines carrying each two flexes.

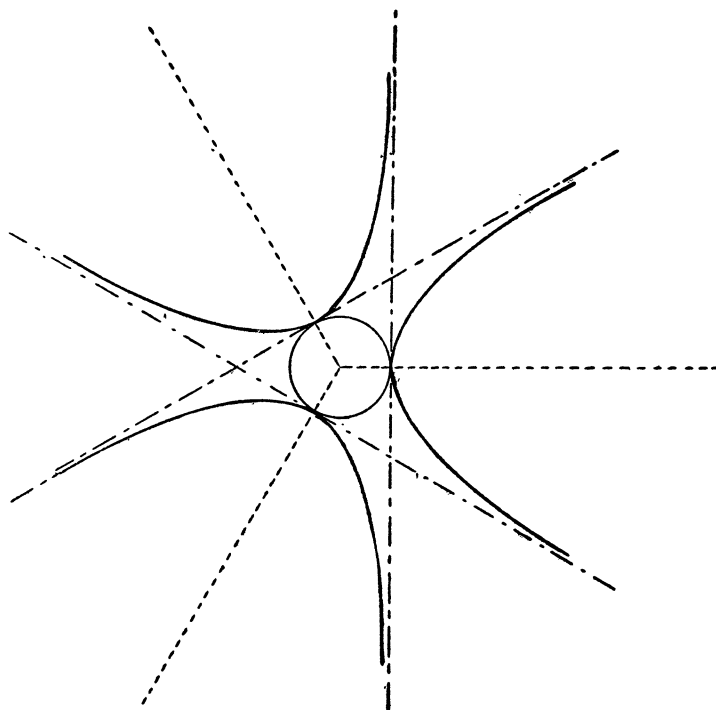


FIG. 3.

The metrical equation of the curve, reflected in the origin, is

$$x = \frac{t^4 - 2t}{t^3 + 1}.$$

The syzygetic line is now the line at infinity, and there are three real inflexional asymptotes (the tangents at the syzygetic points which touch again) and two ordinary ones (the circular rays, imaginary of course). The perspective conic becomes a circle of radius $1/2$, touching ρ^5 where the real asymptotes touch. In other words, the triangle of asymptotes is inscribed in the unit circle and circumscribed about the perspective.

The intersections with the base circle are at the ninth roots of -1 , exclusive of the cube roots. The curve is now readily drawn and a figure is shown above (Fig. 3).

§ 15. *Summary.*

To summarize, we have found, according to our criterion, ten types of self-projective rational quintics, belonging to cyclic groups of orders 2, 3, 4, 5 (two types) and dihedral groups of orders 6 and 10 (two types), besides two with infinite groups.

The parametric equations in canonical form of these as well as of the quartics are shown in the following table:

Self-projective Rational Quartics.

		x_0	x_1	x_2	
cyclic	g_2	$t^4 + at^2,$	$t^2 + 1,$	$t^3 + bt.$	(1)
cyclic	g_3	$t^4,$	$t,$	$t^3 + 1$	(2)
cyclic	g_3	$t^4,$	$t^2,$	$t^3 + 1.$	(3)
cyclic	g_4	$t,$	$t^2,$	$t^4 + 1.$	(4)
dihedral	g_4	$t^3,$	$t,$	$t^4 + at^2 + 1.$	(5)
dihedral	g_6	$at^3 + 1,$	$t^4 + at,$	$t^2.$	(6)
octahedral	g_{24}	$t,$	$t^3,$	$t^4 + 1.$	(7)
	g_∞	$t^4,$	$t,$	$1.$	(8)

Self-projective Rational Quintics.

cyclic	g_2	$t^5 + at^3 + bt,$	$t^4 + ct^2,$	$t^2 + 1.$	(1)
cyclic	g_3	$t^5 + at^2,$	$bt^3 + 1,$	$t^4 + t.$	(2)
cyclic	g_4	$at^5 + t,$	$t^3,$	$t^4 + 1.$	(3)
cyclic	g_5	$t,$	$t^2,$	$t^5 + 1.$	(4)
cyclic	g_5	$t^3,$	$t,$	$t^5 + 1.$	(5)
dihedral	g_6	$t^5 + at^2,$	$at^3 + 1,$	$t^4 + t.$	(6)
dihedral	g_{10}	$t^3,$	$t^2,$	$t^5 + 1.$	(7)
dihedral	g_{10}	$t,$	$t^4,$	$t^5 + 1.$	(8)
	g_∞	$t^5,$	$t^4,$	$1.$	(9)
	g_∞	$t^5,$	$t^3,$	$1.$	(10)